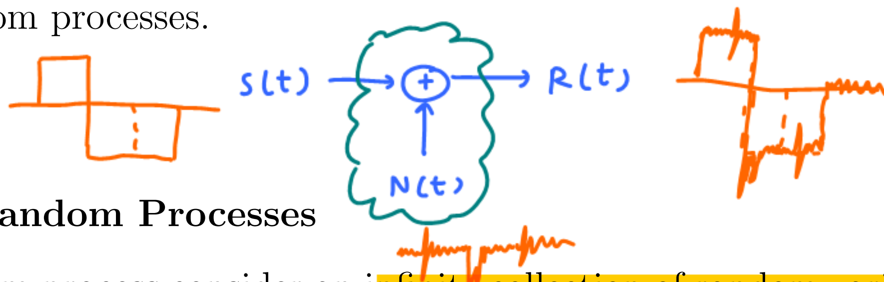
7 The Waveform Channel

The waveform transmitted by the digital demodulator will be corrupted by the channel before it reaches the digital demodulator in the receiver. One important part of the channel is the noise. In continuous time, this random noise is viewed as a random process. So, we first provide some introduction to random processes.



7.1 Random Processes

A random process consider an infinite collection of random variables. These random variables are usually indexed by time. So, the obvious notation for random process would be $X(t)$. As in the signals-and-systems class, time can be discrete or continuous. When time is discrete, it may be more appropriate to use $X_1, X_2, \ldots$ or $X[1], X[2], X[3], \ldots$ to denote a random process.

Example 7.1. Sequence of results (0 or 1) from a sequence of Bernoulli trials is a discrete-time random process.

Example 7.2. Gaussian Random Processes: A random process $X(t)$ is Gaussian if for all positive integers n and for all $t_1, t_2, \ldots, t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian random variables.

Definition 7.3. At any particular time t, because we have a random variable, we can also find its expected value. The function $m_X(t)$ captures these expected values as a deterministic function of time:

1

$$m_X(t) = \mathbb{E}[X(t)].$$

7.1.1 Autocorrelation Function and WSS

One of the most important characteristics of a random process is its autocorrelation function, which leads to the spectral information of the random process. The frequency content process depends on the rapidity of the amplitude change with time. This can be measured by correlating the values of the process at two time instances t_l and t_2.

Definition 7.4. ***Autocorrelation Function***: The autocorrelation function $R_X(t_1, t_2)$ for a random process $X(t)$ is defined by

2

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right].$$

Example 7.5. The random process $x(t)$ is a slowly varying process compared to the process $y(t)$ in Figure 22. For $x(t)$, the values at t_l and t_2 are similar; that is, have stronger correlation. On the other hand, for $y(t)$, values at t_l and t_2 have little resemblance, that is, have weaker correlation.

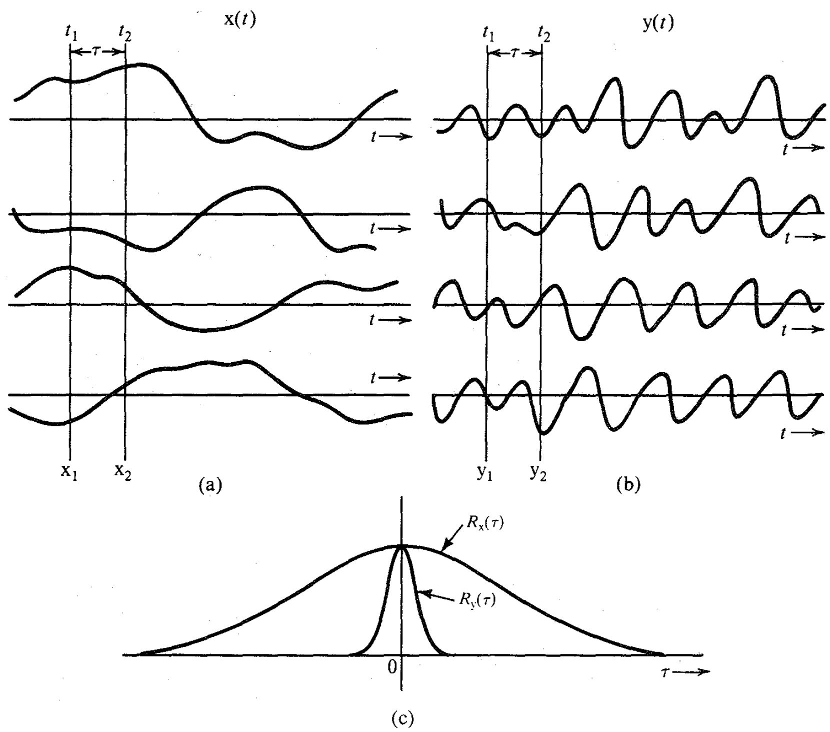


Figure 22: Autocorrelation functions for a slowly varying and a rapidly varying random process [9, Fig. 11.4]

Example 7.6 (Randomly Phased Sinusoid)**.** Consider a random process

$$X(t) = 5\cos(7t + \Theta)$$

where Θ is a uniform random variable on the interval $(0, 2\pi)$.

$$\begin{aligned} m_X(t) = \mathbb{E}\left[X(t)\right] &= \int_{-\infty}^{+\infty} 5\cos(7t+\theta) f_\Theta(\theta) d\theta \\ &= \int_0^{2\pi} 5\cos(7t+\theta)\frac{1}{2\pi} d\theta = 0. \end{aligned}$$

and

$$\begin{aligned}
R_X(t_1,t_2) &= \mathbb{E}\left[X(t_1)X(t_2)\right] \\
&= \mathbb{E}\left[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)\right] \\
&= \frac{25}{2}\cos\left(7(t_2-t_1)\right).
\end{aligned}$$

Definition 7.7. A random process whose statistical characteristics do not change with time is classified as a ***stationary*** random process. For a stationary process, we can say that a shift of time origin will be impossible to detect; the process will appear to be the same.

Example 7.8. The random process representing the temperature of a city is an example of a ***nonstationary*** process, because the temperature statistics (mean value, for example) depend on the time of the day.

On the other hand, the noise process is stationary, because its statistics (the mean ad the mean square values, for example) do not change with time.

7.9. In general, it is not easy to determine whether a process is stationary. In practice, we can ascertain stationary if there is no change in the signal-generating mechanism. Such is the case for the noise process.

A process may not be stationary in the strict sense. A more relaxed condition for stationary can also be considered.

Definition 7.10. A random process $X(t)$ is ***wide-sense stationary (WSS)*** if

(a) $m_X(t)$ is a constant

(b) $R_X(t_1,t_2)$ depends only on the time difference $\overbrace{t_2-t_1}^{\tau}$ and does not depend on the specific values of t_1 and t_2.

In which case, we can write the correlation function as $R_X(\tau)$ where $\tau = t_2 - t_1$.

- One important consequence is that $\mathbb{E}\left[X^2(t)\right]$ will be a constant as well.

Example 7.11. The random process defined in Example 7.5 is WSS with

$$R_X(\tau) = \frac{25}{2}\cos(7\tau).$$

7.12. Most information signals and noise sources encountered in communication systems are well modeled as WSS random processes.

Example 7.13. White noise process is a WSS process $N(t)$ whose

(a) $\mathbb{E}[N(t)] = 0$ for all t and

(b) $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$.

See also 7.19 for its definition.

- Since $R_N(\tau) = 0$ for $\tau \neq 0$, any two different samples of white noise, no matter how close in time they are taken, are uncorrelated.

Example 7.14. [Thermal noise] A statistical analysis of the random motion (by thermal agitation) of electrons shows that the autocorrelation of thermal noise $N(t)$ is well modeled as

$$R_N(\tau) = kTG\frac{e^{-\frac{\tau}{t_0}}}{t_0} \text{ watts},$$

where k is Boltzmann's constant ($k = 1.38 \times 10^{-23}$ joule/degree Kelvin), G is the conductance of the resistor (mhos), T is the (ambient) temperature in degrees Kelvin, and t_0 is the statistical average of time intervals between collisions of free electrons in the resistor, which is on the order of 10^{-12} seconds. [11, p. 105]

3

7.1.2 Power Spectral Density (PSD)

An electrical engineer instinctively thinks of signals and linear systems in terms of their frequency-domain descriptions. Linear systems are characterized by their frequency response (the transfer function), and signals are expressed in terms of the relative amplitudes and phases of their frequency components (the Fourier transform). From the knowledge of the input spectrum and transfer function, the response of a linear system to a given signal can be obtained in terms of the frequency content of that signal. This is an important procedure for deterministic signals. We may wonder if similar methods may be found for random processes.

In the study of stochastic processes, the power spectral density function, $S_X(f)$, provides a frequency-domain representation of the time structure of

$X(t)$. Intuitively, $S_X(f)$ is the expected value of the squared magnitude of the Fourier transform of a sample function of $X(t)$.

You may recall that not all functions of time have Fourier transforms. For many functions that extend over infinite time, the Fourier transform does not exist. Sample functions $x(t)$ of a stationary stochastic process $X(t)$ are usually of this nature. To work with these functions in the frequency domain, we begin with $X_T(t)$, a **truncated** version of $X(t)$. It is identical to $X(t)$ for $-T \le t \le T$ and 0 elsewhere. We use $\mathcal{F}\{X_T\}(f)$ to represent the Fourier transform of $X_T(t)$ evaluated at the frequency f.

Energy : $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{|\underbrace{X(f)}_{\mathcal{F}\{x\}(f)}|^2}_{\text{ESD}} df = \int_{-\infty}^{\infty} \underbrace{|\mathcal{F}\{x\}(f)|^2}_{\text{ESD}} df$

Power : $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\underbrace{x_T(t)}_{\text{Truncated } x(t)}|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\mathcal{F}\{x_T\}(f)|^2 df$ PSD

$x_T(t) = \begin{cases} x(t), & -T < t < T, \\ 0, & \text{otherwise} \end{cases}$

Definition 7.15. Consider a WSS process $X(t)$. The ***power spectral density*** (PSD) is defined as

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[|\mathcal{F}\{X_T\}(f)|^2\right]$$
$$= \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\left|\int_{-T}^{T} X(t) e^{-j2\pi f t} dt\right|^2\right]$$

We refer to $S_X(f)$ as a density function because it can be interpreted as the amount of power in $X(t)$ in the small band of frequencies from f to $f + df$.

7.16. ***Wiener-Khinchine theorem***: the PSD of a WSS random process is the Fourier transform of its autocorrelation function:

$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$R_X(\tau) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} S_X(f)$

and

$$R_X(\tau) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} df.$$

One important consequence is

$$R_X(0) = \mathbb{E}\left[X^2(t)\right] = \int_{-\infty}^{+\infty} S_X(f) df.$$

Example 7.17. For the thermal noise in Example 7.14, the corresponding PSD is $S_N(f) = \frac{2kTG}{1+(2\pi f t_0)^2}$ watts/hertz.

7.18. Observe that the thermal noise's PSD in Example 7.17 is approximately flat over the frequency range 0–10 gigahertz. As far as a typical communication system is concerned we might as well let the spectrum be flat over all frequency, i.e.,

$$S_N(f) = \frac{N_0}{2} \text{ watts/hertz},$$

where N_0 is a constant; in this case $N_0 = 4kTG$.

Definition 7.19. Noise that has a uniform spectrum over the entire frequency range is referred to as **white noise**. In particular, for white noise,

$N(t)$

$\mathbb{E}[N(t)] = 0$

$$S_N(f) = \frac{N_0}{2} \text{ watts/hertz},$$

$\mathcal{F}$

$R_N(\tau) = R_N(t_1, t_2) \equiv \mathbb{E}[N(t_1)N(t_2)]$

$= \frac{N_0}{2}\delta(t_2 - t_1) = \frac{N_0}{2}\delta(\tau)$

- The factor 2 in the denominator is included to indicate that $S_N(f)$ is a two-sided spectrum.
- The adjective "white" comes from white light, which contains equal amounts of all frequencies within the visible band of electromagnetic radiation.
- The average power of white noise is obviously infinite.
  (a) White noise is therefore an abstraction since no physical noise process can truly be white.
  (b) Nonetheless, it is a useful abstraction.

- The noise encountered in many real systems can be assumed to be approximately white.
- This is because we can only observe such noise after it has passed through a real system, which will have a finite bandwidth. Thus, as long as the bandwidth of the noise is significantly larger than that of the system, the noise can be considered to have an infinite bandwidth.
- As a rule of thumb, noise is well modeled as white when its PSD is flat over a frequency band that is 35 times that of the communication system under consideration. [11, p 105]

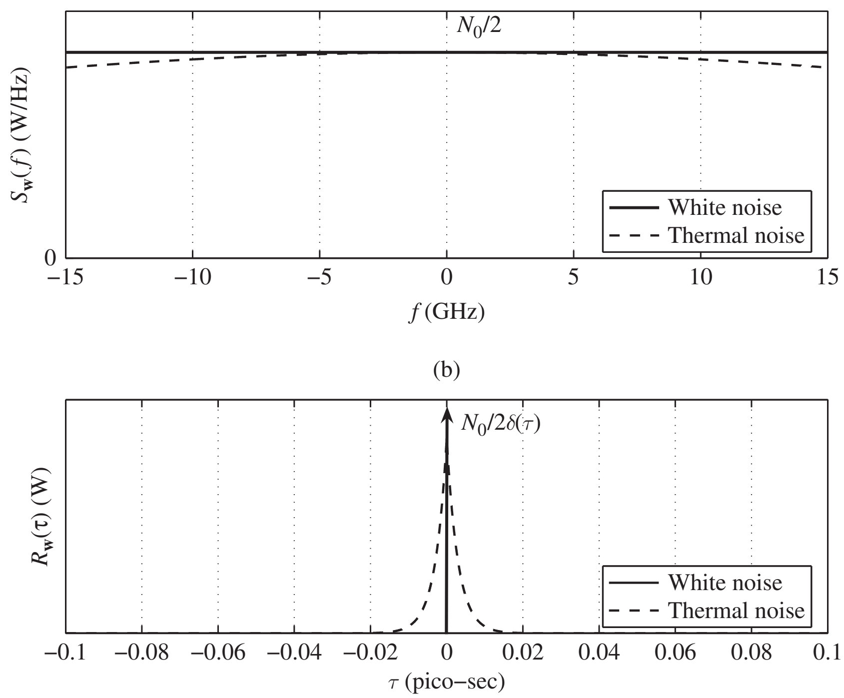


Figure 23: (a) The PSD ($S_N(f)$), and (b) the autocorrelation ($R_N(\tau)$) of noise. (Assume $G = 1/10$ (mhos), $T = 298.15$ K, and $t_0 = 3 \times 10^{-12}$ seconds.) [11, Fig. 3.11]

Theorem 7.20. When we input $X(t)$ through an LTI system whose frequency response is $H(f)$. Then, the PSD of the output $Y(t)$ will be given by

$$S_Y(f) = S_X(f)|H(f)|^2.$$